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著者	YOSHISE Akiko
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by

Akiko YOSHISE

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UNIVERSITY OF TSUKUBA
Tsukuba, Ibaraki 305-8573
JAPAN

Complementarity Problems over Symmetric Cones: A Survey of Recent Developments in Several Aspects

Akiko Yoshise*

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Abstract

The complementarity problem over a symmetric cone (that we call the Symmetric Cone Complementarity Problem, or the SCCP) has received much attention of researchers in the last decade. Many of studies done on the SCCP can be categorized into the three research themes, interior point methods for the SCCP, merit or smoothing function methods for the SCCP, and various properties of the SCCP. In this paper, we will provide a brief survey on the recent developments on these three themes.

1 Introduction

The complementarity problem over a symmetric cone (that we call the *Symmetric Cone Complementarity Problem*, or the *SCCP*) has received much attention of researchers in the last decade. In this chapter, we will provide a brief survey on the recent developments related to the problem.

Let $\mathcal{V}$ be a finite dimensional real vector space with an inner product denoted by $\langle \cdot, \cdot \rangle$ and $\mathcal{K}$ be a symmetric cone in $\mathcal{V}$ which is a closed convex cone with nonempty interior and self-dual, i.e., satisfies

$$\mathcal{K} = \mathcal{K}^* := \{x \in \mathcal{V} \mid \langle x, y \rangle \geq 0 \text{ for all } y \in \mathcal{K}\}. \tag{1}$$

A detailed definition of the symmetric cone will be given in Section 2.

A typical SCCP is the following *standard SCCP* of the form

$$\left| \begin{array}{ll} \text{Find} & (x, y) \in \mathcal{K} \times \mathcal{K} \\ \text{s.t.} & y - \psi(x) = 0, \ x \circ y = 0 \end{array} \right. \tag{2}$$

where $\psi : \Omega \to \mathcal{V}$, Ω is an open domain containing $\mathcal{K}$ and ψ is differentiable on Ω. When the function ψ is affine, we call the problem the *standard linear SCCP*. Many studies have focused on more general problem, the *implicit SCCP*, of the form

$$\left| \begin{array}{ll} \text{Find} & (x, y, z) \in \mathcal{K} \times \mathcal{K} \times \Re^m \\ \text{s.t.} & F(x, y, z) = 0, \ \langle x, y \rangle = 0 \end{array} \right. \tag{3}$$

*Graduate School of Systems and Information Engineering, University of Tsukuba, Tsukuba, Ibaraki 305-8573, Japan. e-mail: yoshise@sk.tsukuba.ac.jp Research supported in part by Grants-in-Aid for Scientific Research ((C)20560052, (C)19510137, (B)18310101, (B)20310082) of the Ministry of Education, Culture, Sports, Science and Technology of Japan.

where $F : \Omega \times \Re^m \to \mathcal{V} \times \Re^m$, Ω in an open domain containing $\mathcal{K} \times \mathcal{K} \times \Re^m$ and F is differentiable on Ω. When the function F is affine, we call the problem the *implicit linear SCCP* given by

$$\left| \begin{array}{ll} \text{Find} & (x, y, z) \in \mathcal{K} \times \mathcal{K} \times \Re^m \\ \text{s.t.} & Px + Qy + Rz - a = 0, \ \langle x, y \rangle = 0 \end{array} \right. \tag{4}$$

where a is a vector in $\mathcal{V} \times \Re^m$, $P : \mathcal{V} \to \mathcal{V} \times \Re^m$, $Q : \mathcal{V} \to \mathcal{V} \times \Re^m$ and $R : \Re^m \to \mathcal{V} \times \Re^m$ are linear operators. Another important special case of the implicit SCCP is so called the *vertical SCCP* of the form

$$\left| \begin{array}{ll} \text{Find} & x \in \mathcal{V} \\ \text{s.t.} & F(x) \in \mathcal{K}, \ G(x) \in \mathcal{K}, \ \langle F(x), G(x) \rangle = 0 \end{array} \right. \tag{5}$$

where $F : \mathcal{V} \to \mathcal{V}$ and $G : \mathcal{V} \to \mathcal{V}$ are differentiable. The vertical SCCP includes the standard SCCP as a special case.

We often assume that the functions associated with the above problems to be monotone. For the implicit SCCP (3), if the function F satisfies

$$\left. \begin{array}{l} (x, y, z), (x', y', z') \in \mathcal{K} \times \mathcal{K} \times \Re^m, \\ F(x, y, z) = F(x', y', z') \end{array} \right\} \implies \langle x - x', y - y' \rangle \geq 0 \tag{6}$$

it is said to be monotone and we call the problem the *monotone implicit SCCP*. The monotone property (6) implies

$$x, x' \in \mathcal{K} \implies \langle F(x) - F(x'), G(x) - G(x') \rangle \geq 0. \tag{7}$$

for the functions F and G of the vertical SCCP (5),

$$x, x' \in \mathcal{K} \implies \langle x - x', \psi(x) - \psi(x') \rangle \geq 0. \tag{8}$$

for ψ of the standard SCCP (2), and

$$Px + Qy + Rz = 0 \implies \langle x, y \rangle \geq 0 \tag{9}$$

for P, Q and R of the implicit linear SCCP (4).

Note that the monotone implicit linear SCCP (4) is a generalization of linear optimization problems over symmetric cones. Consider a primal-dual pair of linear optimization problems over a symmetric cone defined by

$$\begin{array}{lll} (P) & \min & \langle c, x \rangle, \ \text{s.t. } Ax = b, \ x \in \mathcal{K}, \\ (D) & \max & b^T z, \ \text{s.t. } A^T z + y = c, \ y \in \mathcal{K} \end{array}$$

where $A : \mathcal{V} \to \Re^m$ is a linear operator, $b \in \Re^m$ and $c \in \mathcal{V}$. Let α_P and α_D denote the primal and dual optimal objective values, respectively, i.e.,

$$\begin{array}{rcl} \alpha_P & := & \inf\{\langle c, x \rangle \mid Ax = b, \ x \in \mathcal{K}\}, \\ \alpha_D & := & \sup\{b^T z \mid A^T z + y = c, \ y \in \mathcal{K}\}. \end{array}$$

It is known that the following duality theorem holds for the problems (P) and (D) (see Theorems 3.2.6 and 3.2.8 of [105]).

Theorem 1.1 (Duality theorem of the conic optimization). *If the dual problem (D) is strongly feasible (i.e., there exists a $(y, z) \in \Re^m \times \mathcal{V}$ such that $A^T z + y = c$ and $y \in \operatorname{int} K$) and the primal problem (P) is feasible, then the primal problem (P) has an optimal solution. Similarly, if the primal problem (P) is strongly feasible (i.e., there exists an $x \in \mathcal{V}$ such that $Ax = b$ and $x \in \operatorname{int} K$) and the dual problem (D) is feasible, then the dual problem (D) has an optimal solution. In either case, $\alpha_P = \alpha_D$.*

For any primal feasible solution x of (P) and any dual feasible solution (y, z) of (D), we see that

$$\langle c, x\rangle - b^T z = \langle A^T z + y, x\rangle - (Ax)^T z = \langle y, x\rangle \geq 0$$

where the last inequality follows from $x \in \mathcal{K}$, $y \in \mathcal{K}$ and the self-duality (1) of $\mathcal{K}$. Therefore, if we define

$$P := \begin{pmatrix} O \\ A \end{pmatrix}, \quad Q := \begin{pmatrix} I \\ O \end{pmatrix}, \quad R := \begin{pmatrix} A^T \\ O \end{pmatrix}, \quad a := \begin{pmatrix} c \\ b \end{pmatrix},$$

then P, Q and R satisfy

$$Px + Qy + Rz = 0 \implies \langle x, y\rangle = 0$$

which implies that $Px+Qy+Rz$ is monotone and the corresponding monotone implicit linear SCCP (4) is the problem to find an optimal solution of the primal-dual optimization problems over the symmetric cone.

In this chapter, a brief survey of the recent developments on the SCCP will be provided focusing on the following three aspects:

- Interior point methods for the SCCP.
- Merit or smoothing function methods for the SCCP.
- Properties of the SCCP.

After giving a brief introduction to Euclidean Jordan algebras in Section 2, we review some studies placed in the above three categories, respectively, in Sections 3, 4 and 5. We will give some concluding remarks in Section 6.

Before closing this section, we explain some symbols which are used in this chapter. For a given set $\mathcal{S} \subseteq \mathcal{V}$, $\operatorname{int}\mathcal{S}$ and $\operatorname{conv}\mathcal{S}$ denote the interior and the convex hull of $\mathcal{S}$, respectively.

2 Euclidean Jordan algebra

In this section, we give a brief introduction to Euclidean Jordan algebras. See the chapter by Farid Alizadeh in this handbook or the monograph by Faraut and Korányi [27] for a more comprehensive introduction.

A Euclidean Jordan algebra is a triple $(\mathcal{V}, \circ, \langle\cdot,\cdot\rangle)$ where $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is a n-dimensional inner product space over $\Re$ and $(x, y) \mapsto x \circ y$ on $\mathcal{V}$ is a bilinear mapping which satisfies the following conditions for all $x, y \in \mathcal{V}$:

$$\begin{cases} \text{(i)} & x \circ y = y \circ x, \\ \text{(ii)} & x \circ (x^2 \circ y) = x^2 \circ (x \circ y) \quad \text{where } x^2 = x \circ x, \\ \text{(iii)} & \langle x \circ y, z\rangle = \langle x, y \circ z\rangle. \end{cases} \tag{10}$$

We call $x \circ y$ the *Jordan product* of x and y. Note that $(x \circ y) \circ w \neq x \circ (y \circ w)$ in general. We assume that there exists an element e (called as the *identity element*) such that $x \circ e = e \circ x = x$ for all $x \in \mathcal{V}$.

The *rank* of $(\mathcal{V}, \circ, \langle\cdot,\cdot\rangle)$ is defined as

$$r := \max\{\deg(x) \mid x \in \mathcal{V}\}$$

where $\deg(x)$ is the degree of $x \in \mathcal{V}$ given by

$$\deg(x) := \min\left\{k \mid \{e, x, x^2, \ldots, x^k\} \text{ are linearly dependent}\right\}.$$

The *symmetric cone* $\mathcal{K}$ is a self-dual (i.e., $\mathcal{K}$ satisfies (1)) closed convex cone with nonempty interior and homogeneous (i.e., for all $x, y \in \operatorname{int}\mathcal{K}$, there exists a invertible linear map G which satisfies $G(x) = y$ and $G(\mathcal{K}) = \mathcal{K}$). By Theorem III 2.1 in [27], the symmetric cone $\mathcal{K}$ coincides with the *set of squares* $\{x^2 \mid x \in \mathcal{V}\}$ of some Euclidean Jordan algebra $\mathcal{V}$.

For any $x \in \mathcal{V}$, the *Lyapunov transformation* $L_x : \mathcal{V} \to \mathcal{V}$ is defined as $L_x y = x \circ y$ for all $y \in \mathcal{V}$. It follows from (i) and (iii) of (10) that the Lyapunov transformation is symmetric, i.e., $\langle L_x y, z\rangle = \langle y, L_x z\rangle$ holds for all $y, z \in \mathcal{V}$. Especially, $L_x e = x$ and $L_x x = x^2$ hold for all $x \in \mathcal{V}$. Using the Lyapunov transformation, the *quadratic representation* of $x \in \mathcal{V}$ is defined as

$$Q_x = 2L_x^2 - L_{x^2}. \tag{11}$$

For any $x \in \mathcal{K}$, L_x is positive semidefinite and it holds that

$$\langle x, y\rangle = 0 \iff x \circ y = 0 \tag{12}$$

for any $x, y \in \mathcal{K}$ (see Lemma 8.3.5 of [1]).

An element $c \in \mathcal{V}$ is an *idempotent* if $c^2 = c \neq 0$, which is also *primitive* if it cannot be written as a sum of two idempotents. A *complete system of orthogonal idempotents* is a finite set $\{c_1, c_2, \cdots, c_k\}$ of idempotents where $c_i \circ c_j = 0$ for all $i \neq j$, and $c_1 + c_2 + \cdots + c_k = e$. A *Jordan frame* is a complete system of orthogonal primitive idempotents in $\mathcal{V}$. The following theorem gives us a *spectral decomposition* for the elements in a Euclidean Jordan algebra (see Theorem III.1.2 of [27]).

Theorem 2.1 (Spectral decomposition theorem)**.** *Let $(\mathcal{V}, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra with rank r. Then for any $x \in \mathcal{V}$, there exist a Jordan frame $\{c_1, c_2, \ldots, c_r\}$ and real numbers $\lambda_i(x)$ $(i = 1, 2, \ldots, r)$ such that*

$$x = \sum_{i=1}^{r} \lambda_i(x) c_i.$$

The numbers $\lambda_i(x)$ $(i = 1, 2, \ldots, r)$ are called the eigenvalues *of x, which are uniquely determined by x.*

Note that the Jordan frame in the above theorem depends on x, but we omit the dependence in order to simplify the notation. If two elements x and y share the same Jordan frames in the decompositions in Theorem 2.1 then they *operator commute*, i.e., they satisfy $L_x L_y = L_y L_x$. An eigenvalue $\lambda_i(x)$ is continuous with respect to x. The *trace* of x is defined as $\operatorname{tr}(x) = \sum_{i=1}^{r} \lambda_i$ and the *determinant* of x is defined as $\det(x) = \prod_{i=1}^{r} \lambda_i$. We also see that $x \in \mathcal{K}$ (respectively, $x \in \operatorname{int}\mathcal{K}$) if and only if $\lambda_i(x) \geq 0$ (respectively, $\lambda_i(x) > 0$) for all $i = 1, 2, \ldots, r$. For any $x \in \mathcal{V}$ having the spectral decomposition $x = \sum_{i=1}^{r} \lambda_i(x) c_i$, we denote

$$x^{1/2} = \sqrt{x} := \sum_{i=1}^{r} \sqrt{\lambda_i(x)}\, c_i \quad \text{if } \lambda_i(x) \geq 0 \text{ for all } i = 1, 2, \ldots, r,$$

$$x^{-1} := \sum_{i=1}^{r} \lambda_i(x)^{-1} c_i \quad \text{if } \lambda_i(x) \neq 0 \text{ for all } i = 1, 2, \ldots, r.$$

Now we introduce another decomposition, the *Peirce decomposition*, on the space $\mathcal{V}$ (see Theorem IV.2.1 of [27]).

Theorem 2.2 (Peirce decomposition theorem)**.** *Let $\{c_1, \dots, c_r\}$ be a Jordan frame. Define*

$$\begin{aligned} \mathcal{V}_i &:= \{\theta c_i \mid \theta \in \Re\}, \\ \mathcal{V}_{ij} &:= \left\{ x \in \mathcal{V} \mid c_i \circ x = \frac{1}{2} x = c_j \circ x \right\} \quad (i < j). \end{aligned}$$

Then for any $x \in \mathcal{V}$, there exists $x_i \in \Re$, $c_i \in \mathcal{V}_i$ and $x_{ij} \in \mathcal{V}_{ij}$ $(i < j)$,

$$x = \sum_{i=1}^{r} x_i c_i + \sum_{i<j} x_{ij}.$$

Fix a Jordan frame $\{c_1, c_2, \dots, c_l\}$ and define

$$\mathcal{V}^{(l)} := \{x \in \mathcal{V} \mid x \circ (c_1 + c_2 + \cdots + c_l) = x\}$$

for $1 \le l \le r$. Corresponding to $\mathcal{V}^{(l)}$, we consider the (orthogonal) projection $P^{(l)} : \mathcal{V} \to \mathcal{V}^{(l)}$. For a given linear transformation $L : \mathcal{V} \to \mathcal{V}$, we denote $L_{\{c_1, c_2, \dots, c_l\}}$ the composite transformation $P^{(l)} \bullet L : \mathcal{V} \to \mathcal{V}^{(l)}$. We call $L_{\{c_1, c_2, \dots, c_l\}}$ the *principal subtransformation* of L corresponding to $\{c_1, c_2, \dots, c_l\}$ and the determinant of $L_{\{c_1, c_2, \dots, c_l\}}$ a *principal minor* of L.

Typical examples of Euclidean Jordan algebras are

(i) *Euclidean Jordan algebra of n-dimensional vectors* where

$$\mathcal{V} = \Re^n, \quad \mathcal{K} = \Re^n_+ := \{x \in \Re^n \mid x \ge 0\},$$

(ii) *Euclidean Jordan algebra of n-dimensional symmetric matrices* where

$$\mathcal{V} = \mathcal{S}^n := \{X \in \Re^{n \times n} \mid X = X^T\}, \ \mathcal{K} = \mathcal{S}^n_+ := \{X \in \mathcal{S}^n \mid X \succeq O\}$$

and $X \succeq O$ denotes that X is a positive semidefinite matrix, and

(iii) *Euclidean Jordan algebra of quadratic forms* where

$$\mathcal{V} = \Re^n, \ \mathcal{K} = \mathcal{L}^n := \{(x_1, x_2) \in \Re \times \Re^{n-1} \mid \|x_2\| \le x_1\}$$

and $\| \cdot \|$ denotes the Euclidean norm.

In this chapter, we call the SCCP the *nonlinear complementarity problem* (NCP) when $\mathcal{V}$ and $\mathcal{K}$ are given in (i), the *semidefinite complementarity problem* (SDCP) when $\mathcal{V}$ and $\mathcal{K}$ are given in (ii) and the *second order cone complementarity problem* (SOCCP) when $\mathcal{V}$ and $\mathcal{K}$ are given in (iii), respectively.

3 Interior-point methods for the SCCP

The first interior point algorithm for solving the SCCP has been provided in Chapter 7 of Nesterov and Nemirovski's seminal book [88] while the connection to the Euclidean Jordan algebra has not been pointed out clearly. The algorithm is an interior point algorithm based on a *self-concordant barrier* only in the variables x, which is closely related to symmetric cones [43]. The polynomial complexity bound and a way to find an appropriate initial point have been discussed.

After about four years from the publication of [88], in 1997, a first interior point algorithm in the setting of a Euclidean Jordan algebra has been given by Faybusovich [28] which employs the barrier function in x and y of the form

$$\mu\langle x, y\rangle - \log\det(x) - \log\det(y).$$

Independently of Faybusovich's work, an interior point algorithm for solving the n-dimensional monotone implicit linear SDCP (4) (called as the *monotone implicit SDLCP*) has been provided by Kojima, Shindoh and Hara [53]. In the paper, after showing that the convex quadratic semidefinite optimization problem can be cast both into a monotone SDLCP and into a semidefinite optimization problem (called as the *SDP*), the authors have mentioned that

> This fact itself never denies the significance of the monotone SDLCP because the direct SDLCP formulation is of a smaller size than the SDP formulation but raises questions like how general the monotone SDLCP is and whether it is essentially different from the semidefinite optimization problem. In their recent paper [54], Kojima, Shida and Shindoh showed that the monotone SDLCP is reducible to a SDP involving an additional m-dimensional variable vector and an $(m+1)\times(m+1)$ variable symmetric matrix, where $m = n(n+1)/2$.

The convex cone $\mathcal{K}$ considered in [54] is quite general, i.e., $\mathcal{K}$ is just nonempty closed and convex. Therefore, the raised questions in [53] are also the questions to the monotone implicit linear SCCP (4).

In fact, many results on the primal-dual interior point algorithms for solving the SDP can be easily extended to solve the monotone implicit linear SDCP (4) (i.e., monotone implicit SDLCP), and then the extended results can be further extended to solve the monotone implicit linear SCCP (4) using the fundamental results established by Alizadeh and Schmieta [1, 107] for the primal and dual symmetric cone linear optimization problems.

This may be a reason why there are far fewer papers on interior point algorithms for solving the monotone implicit linear SCCP (4) than those for solving the primal and dual symmetric cone linear optimization problems.

Let us consider a more general problem, the monotone implicit SCCP (3) satisfying (6). In view of the property (12), the problem (3) is equivalent to find a $(x, y, z) \in \mathcal{K}\times\mathcal{K}\times\Re^m$, i.e., it satisfies

$$x \circ y = 0, \quad F(x, y, z) = 0. \tag{13}$$

Any interior point algorithm for solving the problem generates a sequence $\{(x_k, y_k, z_k)\} \subset \mathcal{K}\times\mathcal{K}\times\Re^m$ satisfying $x_k \in \operatorname{int}\mathcal{K}$ and $y_k \in \operatorname{int}\mathcal{K}$ $(k = 1, 2, \ldots)$. Let $(x_0, y_0, z_0) \in \operatorname{int}\mathcal{K}\times\operatorname{int}\mathcal{K}\times\Re^m$ be an arbitrary initial point. It is not too often that the initial point (x_0, y_0, z_0) is a *feasible interior point* of (3) such that $F(x_0, y_0, z_0) = 0$. So in practice, we have to assume that $F(x_0, y_0, z_0) \neq 0$. Obviously, the system

$$x \circ y = x_0 \circ y_0, \quad F(x, y, z) = F(x_0, y_0, z_0) \tag{14}$$

has a trivial solution (x_0, y_0, z_0) while the target system is given by (13). Let $H : \operatorname{int}\mathcal{K}\times\operatorname{int}\mathcal{K}\times\Re^m \to \mathcal{V}\times\mathcal{V}\times\Re^m$ be the so-called *interior point map* which is defined as

$$H(x, y, z) := (x \circ y,\ F(x, y, z)). \tag{15}$$

Then the systems (14) and (13) can be represented using H as

$$H(x, y, z) = (x_0 \circ y_0,\ F(x_0, y_0, z_0)) \tag{16}$$

and

$$H(x, y, z) = (0,\ 0), \tag{17}$$

respectively. Introducing a parameter $\mu \in (0, 1]$, consider the system

$$H(x, y, z) = \mu \left(x_0 \circ y_0,\ F(x_0, y_0, z_0) \right). \tag{18}$$

The system has a trivial solution when $\mu = 1$. If the system (18) has the unique solution $(x(\mu), y(\mu), z(\mu))$ for each $\mu \in (0, 1]$ and $(x(\mu), y(\mu), z(\mu))$ is continuous at $\mu \in (0, 1)$, then we may numerically trace the path $\{(x(\mu), y(\mu)), z(\mu)\}$ from the initial point (x_0, y_0, z_0) to a solution of the monotone implicit SCCP (3). This is a basic idea of the (infeasible) interior point algorithm for the SCCP.

Most of studies related to the interior point algorithms for the SCCP have dealt with one of the following subjects:

- The interior point map and its properties.
- Algorithms and their convergence properties.

In what follows, we observe some recent results on the above subjects.

3.1 Interior point map and its properties

It is important to observe the properties of the interior point map H of (15) to find whether the system (18) has the unique solution $(x(\mu), y(\mu), z(\mu))$ for each $\mu \in (0, 1]$. If it holds then the set $\{(x(\mu), y(\mu), z(\mu)) \mid \mu \in (0, 1]\}$ will give us an *(infeasible) interior point trajectory* of the SCCP.

For the monotone SDCP, Shida, Shindoh and Kojima [108] investigated the existence and continuity of the (infeasible) interior point trajectory in a general setting of the problem. On the other hand, Monteiro [83] showed a vast set of conclusions concerning the interior point map for the monotone SDCP. Based on the paper [83], the following results have been shown in [127] for the monotone implicit SCCP (3) satisfying (6).

Let $\mathcal{U}$ be a subset of $\mathrm{int}\,\mathcal{K} \times \mathrm{int}\,\mathcal{K}$ defined by

$$\mathcal{U} := \{(x, y) \in \mathrm{int}\,\mathcal{K} \times \mathrm{int}\,\mathcal{K} :\ x \circ y \in \mathrm{int}\,\mathcal{K}\}.$$

The following assumption has been imposed in [127].

Assumption 3.1. **(i)** *$F : \mathcal{K} \times \mathcal{K} \times \Re^m \to \mathcal{V} \times \Re^m$ is monotone on its domain, i.e., F satisfies (6).*

(ii) *$F : \mathcal{K} \times \mathcal{K} \times \Re^m \to \mathcal{V} \times \Re^m$ is z-bounded on its domain, i.e., for any sequence $\{(x_k, y_k, z_k)\}$ in the domain of F, if $\{((x_k, y_k)\}$ and $\{F(x_k, y_k, z_k)\}$ are bounded then the sequence $\{z_k\}$ is also bounded.*

(iii) *$F : \mathcal{K} \times \mathcal{K} \times \Re^m \to \mathcal{V} \times \Re^m$ is z-injective on its domain, i.e., if (x, y, z) and (x, y, z') lie in the domain of F and satisfy $F(x, y, z) = F(x, y, z')$, then $z = z'$ holds.*

Under the assumption above, the homeomorphism of the interior point map H has been shown (see Theorem 3.10 of [127]).

Theorem 3.2 (Homeomorphism of the interior point map)**.** *Suppose that a continuous map $F : \mathcal{K} \times \mathcal{K} \times \Re^m \to \mathcal{V} \times \Re^m$ satisfies Assumption 3.1. Then the map H defined by (15) maps $\mathcal{U} \times \Re^m$ homeomorphically onto $\mathcal{K} \times F(\mathcal{U} \times \Re^m)$.*

The theorem above ensures that if the monotone implicit SCCP (3) has an interior feasible point $(\bar{x}, \bar{y}, \bar{z}) \in \operatorname{int} \mathcal{K} \times \operatorname{int} \mathcal{K} \times \Re^m$ which satisfies

$$\bar{x} \circ \bar{y} \in \operatorname{int} \mathcal{K} \text{ and } F(\bar{x}, \bar{y}, \bar{z}) = 0$$

and if we have a bounded path $\{p(\mu) \mid \ \mu \in [0, 1]\} \subseteq \mathcal{K} \times F(\mathcal{U} \cap \Re^m)$ such that

$$p(0) = 0 \ \text{ and } \ p(\mu) \in \operatorname{int} \mathcal{K} \times F(\mathcal{U} \times \Re^m)$$

then there exists a unique path $\{(x(\mu), y(\mu), z(\mu)) \mid \mu \in (0, 1]\} \subseteq \operatorname{int} \mathcal{K} \times \operatorname{int} \mathcal{K} \times \Re^m$ for which

$$H(x(\mu), y(\mu), z(\mu)) = p(\mu) \text{ for all } \mu \in (0, 1]$$

holds and whose any accumulation point is a solution of the monotone implicit SCCP (3). Thus the path $\{(x(\mu), y(\mu), z(\mu)) \mid \mu \in (0, 1]\}$ is an *(infeasible) interior point trajectory* (see Corollary 4.4 of [127]). In addition, a condition on F so that we can take

$$p(\mu) = \mu\,(x_0 \circ y_0,\ F(x_0, y_0, z_0))$$

as in (18) has been provided in [127]. Note that if $x_0 \circ y_0 = e$, we sometimes call the (infeasible) interior point trajectory the *(infeasible) central trajectory* or *(infeasible) central path.*

3.2 Algorithms and their convergence properties

In what follows, we will describe outlines of several interior point algorithms which are based on tracing the (infeasible) interior point trajectory $\{(x(\mu), y(\mu), z(\mu)) \mid \mu \in (0, 1]\}$.

3.2.1 Infeasible interior point algorithm

Potra [101] proposed an infeasible interior point algorithm for solving the monotone implicit linear SCCP (4). The algorithm is a generalization of the *corrector-predictor approach* proposed in [41, 100]. The outline of the algorithm is as follows.

Let (μ_0, x_0, y_0, z_0) be a starting point satisfying $\mu_0 = 1$ and $x_0 \circ y_0 = e$. At each iteration (x, y, z), the target system is given by

$$H(x, y, z) = \mu_k\,(x_0 \circ y_0,\ 0) = \mu_k\,(e,\ 0) \tag{19}$$

where $\mu_k \in (0, \mu_0]$. Applying Newton's method to the system (19) at (x, y, z) leads us to the linear system

$$\begin{array}{l} y \circ \Delta x + \Delta y \circ x = \gamma \mu_k e - x \circ y, \\ P\Delta x + Q\Delta y + R\Delta z = (1 - \gamma)(a - Px - Qy - Rz) \end{array} \tag{20}$$

where $\gamma \in [0, 1]$ is a parameter for regulating the feasibility and the complementarity.

Let us choose a $p \in \operatorname{int} \mathcal{K}$ belonging to the *commutative class of scalings* for (x, y),

$$\mathcal{C}(x, y) = \{p \in \operatorname{int} \mathcal{K} \mid Q_p x \text{ and } Q_{p^{-1}} y \text{ operator commute}\} \tag{21}$$

where Q_p is the quadratic representation of p defined by (11) and consider the scaled quantities

$$\tilde{x} = Q_p x,\ \ \tilde{y} = Q_{p^{-1}} y,\ \ \tilde{P} = P Q_{p^{-1}},\ \ \tilde{Q} = Q Q_p.$$

Since $\tilde{x} \circ \tilde{y} = \mu_k e$ if and only if $x \circ y = \mu_k e$, the target system (19) can be written under the form

$$\tilde{x} \circ \tilde{y} = \mu_k e, \quad \tilde{P}\tilde{x} + \tilde{Q}\tilde{y} + Rz = a$$

and the Newton system at $(\tilde{x}, \tilde{y}, z)$ is given by

$$\begin{aligned} &\tilde{y} \circ \tilde{\Delta x} + \tilde{\Delta y} \circ \tilde{x} = \gamma\mu_k e - \tilde{x} \circ \tilde{y}, \\ &\tilde{P}\tilde{\Delta x} + \tilde{Q}\tilde{\Delta y} + R\Delta z = (1-\gamma)(a - Px - Qy - Rz). \end{aligned}$$

Since we have chosen p to be in the the commutative class of scalings $\mathcal{C}(x,y)$, the resulting $\tilde{x}$ and $\tilde{y}$ share the same Jordan frame, the above system has a unique solution $(\tilde{\Delta x}, \tilde{\Delta y}, \Delta z)$ and we obtain a *commutative class of the search directions* $(\Delta x, \Delta y, \Delta z)$ by

$$(\Delta x, \Delta y, \Delta z) := (Q_p^{-1}\tilde{\Delta x}, Q_{p^{-1}}^{-1}\tilde{\Delta y}, \Delta z)$$

which satisfies (20). The commutative class of the search directions is a subclass of Monteiro and Zhang family (see [84, 86]) and well used in many interior point algorithms for solving optimization problems over the symmetric cones.

The algorithm employs the following *neighborhood of the infeasible central path*

$$\mathcal{N}(\beta) = \{(\mu, x, y, z) \in (0, \mu_0] \times \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K} \times \Re^m \mid \sigma(x,y,z) \subset [\beta\mu, (1/\beta)\mu]\} \tag{22}$$

where a $\beta \in (0,1)$ is a given parameter, and $\sigma(x,y,z)$ denotes the set of all eigenvalues of $Q_{x^{1/2}}y$ with $Q_{x^{1/2}}$ being the quadratic representation of $x^{1/2}$ (see [104]). Let $\lambda_{\min}(x,y,z)$ and $\lambda_{\max}(x,y,z)$ be the minimum and maximum eigenvalues in $\sigma(x,y,z)$, respectively. By introducing the proximity measure

$$\delta(\mu, x, y, z) = \max\left\{1 - \frac{\lambda_{\min}(x,y,z)}{\mu}, 1 - \frac{\mu}{\lambda_{\max}(x,y,z)}\right\} \tag{23}$$

the neighborhood (22) can be written as

$$\mathcal{N}(\beta) = \{(\mu, x, y, z) \in (0, \mu_0] \times \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K} \times \Re^m \mid \delta(\mu, x, y, z) \le 1 - \beta\}.$$

Each iteration of the algorithm consists of two steps, a corrector step and a predictor step.

The intent of the *corrector step* is to increase proximity to the central path. We choose $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ and $\eta = 1 - \gamma$ where $0 < \underline{\gamma} < \bar{\gamma} < 1$ are given parameters, compute the search directions $(\Delta x, \Delta y, \Delta z)$ at (x,y,z), find

$$\alpha_c := \arg\min\left\{\delta\left(\mu, x + \alpha\Delta x, y + \alpha\Delta y, z + \alpha\Delta z\right)\right\},$$

set $(x,y,z) \leftarrow (x,y,z) + \alpha_c(\Delta x, \Delta y, \Delta z)$ and proceed to the predictor step.

In contrast, the intent of the *predictor step* is to decrease the complementarity gap as much as possible while keeping the iterate in $\mathcal{N}(\beta)$. We choose $\gamma = 0$ and $\eta = 1$, compute the search directions $(\Delta x, \Delta y, \Delta z)$ at (x,y,z), find

$$\alpha_p := \max\left\{\bar{\alpha} \mid (x,y,z) + \alpha(\Delta x, \Delta y, \Delta z) \in \mathcal{N}(\beta) \text{ for each } \alpha \in [0, \bar{\alpha}]\right\},$$

set $(x,y,z) \leftarrow (x,y,z) + \alpha_p(\Delta x, \Delta y, \Delta z)$ and proceed to the next iterate.

While the detailed proof has not been proposed, the author asserted that the algorithm generates a sequence $\{(\mu_k, x_k, y_k, y_k)\}$ satisfying $(x_k, y_k, y_k) \in \mathcal{N}(\beta)$ and the polynomial convergence can be obtained under general assumptions using general theoretical results from [107] and [104]. It has been also asserted that the superlinear convergence is proved if the problem has a strictly complementary solution and the sequence $\{(x_k, y_k, y_k)\}$ satisfies $(x_k \circ y_k) / \sqrt{\langle x_k, y_k\rangle} \to 0$.

There are several papers on infeasible interior point algorithms for solving the monotone and linear SDCP, e.g., [53, 98].

3.2.2 Homogeneous algorithm

Another approach to overcome the difficulty to find a feasible interior starting point is the homogeneous algorithm proposed in [127] and [128] for solving the monotone standard SCCP (2) satisfying (8), which is a generalization of the homogeneous algorithm in [2] for the classical symmetric cone $\mathcal{K} = \Re^n_+$.

The homogeneous algorithm is an infeasible interior point algorithm for the *homogeneous model* given by

$$\begin{array}{lll} \text{(HCP)} & \text{Find} & (x, \tau, y, \kappa) \in (\mathcal{K} \times \Re_{++}) \times (\mathcal{K} \times \Re_+) \\ & \text{s.t.} & F_{\mathrm{H}}(x, \tau, y, \kappa) = 0, \ (x, \tau) \circ_{\mathrm{H}} (y, \kappa) = 0 \end{array} \tag{24}$$

where $F_{\mathrm{H}} : (\mathcal{K} \times \Re_{++}) \times (\mathcal{K} \times \Re_+) \to (\mathcal{V} \times \Re)$ and $(x, \tau) \circ_{\mathrm{H}} (y, \kappa)$ are defined as

$$F_{\mathrm{H}}(x, \tau, y, \kappa) := (y, \ \kappa) - \psi_{\mathrm{H}}(x, \tau), \quad \psi_{\mathrm{H}}(x, \tau) := (\tau\psi(x/\tau), \ -\langle \psi(x/\tau), x\rangle) \tag{25}$$

and

$$(x, \tau) \circ_{\mathrm{H}} (y, \kappa) := (x \circ y, \ \tau\kappa) \,. \tag{26}$$

The function ψ_{H} has the following property (see Proposition 5.3 of [127]).

Proposition 3.3 (Monotonicity of the homogeneous function ψ_{H})**.** *If $\psi : \mathcal{K} \to \mathcal{V}$ is monotone, i.e., satisfies (8), then the function ψ_{H} is monotone on* $\operatorname{int}\mathcal{K} \times \Re_{++}$.

For ease of notation, we use the following symbols

$$\mathcal{V}_{\mathrm{H}} := \mathcal{V} \times \Re, \quad \mathcal{K}_{\mathrm{H}} := \mathcal{K} \times \Re_+, \quad x_{\mathrm{H}} := (x, \tau) \in \mathcal{V}_{\mathrm{H}}, \quad y_{\mathrm{H}} := (y, \kappa) \in \mathcal{V}_{\mathrm{H}}, \quad e_{\mathrm{H}} := (e, 1).$$

Note that the set $\mathcal{K}_{\mathrm{H}}$ is a Cartesian product of two symmetric cones $\mathcal{K}$ and $\Re_+$ given by

$$\mathcal{K}_{\mathrm{H}} = \left\{ x_{\mathrm{H}}^2 = \left(x^2, \ \tau^2\right) : \ x_{\mathrm{H}} \in \mathcal{V}_{\mathrm{H}} \right\}$$

which implies that $\mathcal{K}_{\mathrm{H}}$ is the symmetric cone of $\mathcal{V}_{\mathrm{H}}$. It can be easily seen that $\operatorname{int}\mathcal{K}_{\mathrm{H}} = \operatorname{int}\mathcal{K} \times \Re_{++}$.

A merit of the homogeneous model is that it can provide certifications on strong feasibility or strong infeasibility of the original problem by adding the new variables τ and κ. The following theorem has been shown (see Theorem 5.4 of [127]).

Theorem 3.4 (Properties of the homogeneous model)**. (i)** *The HCP (24) is* asymptotically feasible*, i.e., there exists a bounded sequence $\{(x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)})\} \subseteq \operatorname{int}\mathcal{K}_{\mathrm{H}} \times \operatorname{int}\mathcal{K}_{\mathrm{H}}$ such that*

$$\lim_{k\to\infty} F_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}, y_{\mathrm{H}}^{(k)}) = \lim_{k\to\infty} \left(y_{\mathrm{H}} - \psi_{\mathrm{H}}(x_{\mathrm{H}}^{(k)}) \right) = 0.$$

(ii) *the monotone standard SCCP (2) has a solution if and only if the HCP (24) has an asymptotic solution $(x_{\mathrm{H}}^*, y_{\mathrm{H}}^*) = (x^*, \tau^*, y^*, \kappa^*)$ with $\tau^* > 0$. In this case, $(x^*/\tau^*, y^*/\tau^*)$ is a solution of the monotone standard SCCP (2).*

(iii) *Suppose that ψ satisfies the Lipschitz condition on $\mathcal{K}$, i.e., there exists a constant $\gamma \geq 0$ such that*

$$\|\psi(x + h) - \psi(x)\| \leq \gamma\|h\| \ \text{ for any } \ x \in \mathcal{K} \text{ and } h \in \mathcal{V} \text{ such that } x + h \in \mathcal{K}.$$

If the monotone standard SCCP (2) is strongly infeasible*, i.e., there is no sequence $\{x^{(k)}, y^{(k)}\} \subseteq \operatorname{int} K \times \operatorname{int} K$ such that $\lim_{k\to\infty} \left(y^{(k)} - \psi(x^{(k)})\right) = 0$, then the HCP (24) has an asymptotic solution $(x^*, \tau^*, y^*, \kappa^*)$ with $\kappa^* > 0$. Conversely, if the HCP (24) has an asymptotic solution $(x^*, \tau^*, y^*, \kappa^*)$ with $\kappa^* > 0$ then the monotone standard SCCP (2) is infeasible. In the latter case, $(x^*/\kappa^*, y^*/\kappa^*)$ is a certificate to prove infeasibility of the monotone standard SCCP (2).*

Unfortunately, the homogeneous function $\psi_{\rm H}$ undermines linearity of the original function ψ if it has. Let us consider a simple example, $\psi : \Re \to \Re$, $\psi(x) = x$. Then the induced homogeneous function $\psi_{\rm H} : \Re \times \Re_{++} \to \Re^2$ is given by $\psi_{\rm H}(x,\tau) = (x, -x^2/\tau)$. The function $\psi_{\rm H}$ is no longer linear, but $\psi_{\rm H}$ is monotone as in Theorem 3.3 and satisfies the following assumption which can be considered as a scaled Lipschitz condition.

Assumption 3.5. *There exists a $\theta \geq 0$ such that*

$$\left\| \tilde{z}(\alpha) \circ \left(\widetilde{\psi}(\tilde{z}(\alpha)) - \widetilde{\psi}(\tilde{z}) - \alpha D\widetilde{\psi}(\tilde{z})\widetilde{\Delta z} \right) \right\| \leq \alpha^2 \theta \langle \widetilde{\Delta z}, D\widetilde{\psi}(\tilde{z})\widetilde{\Delta z} \rangle$$

for all $z \in \operatorname{int} \mathcal{K}$, $\Delta z \in \mathcal{V}$, $p \in \mathcal{C}(x,y)$ and $\alpha \in [0,1]$ such that $z(\alpha) \in \operatorname{int} \mathcal{K}$, where

$$\tilde{z}(\alpha) = Q_p(z + \alpha \Delta z), \;\; \widetilde{\psi}(\tilde{z}) = Q_p^{-1} \bullet \psi \bullet Q_p^{-1}(\tilde{z}) = Q_p^{-1}\psi(z).$$

Here, $\phi_1 \bullet \phi_2$ denotes the composite function of ϕ_1 and ϕ_2.

Obviously, if ψ is affine then ψ satisfies the assumption with $\theta = 0$.

The homogeneous algorithm in [128] is an infeasible interior point to the homogeneous model (24). It employs the commutative class of search directions (see (21)) including the *yx-direction* (respectively, *xy-direction*) with $p = y^{1/2}$ (respectively, $p = x^{-1/2}$) and the *Nesterov-Todd (NT) direction* with

$$p = \left[Q_{x^{1/2}} (Q_{x^{1/2}} y)^{-1/2} \right]^{-1/2} = \left[Q_{y^{-1/2}} (Q_{y^{-1/2}} x)^{1/2} \right]^{-1/2}.$$

so that $\tilde{x} = \tilde{y}$, and the following three types of neighborhoods

$$\begin{array}{lcl} \mathcal{N}_{\rm F}(\beta) & := & \{(x_{\rm H}, y_{\rm H}) \in \mathcal{K}_{\rm H} \times \mathcal{K}_{\rm H} \mid d_{\rm F}(x_{\rm H}, y_{\rm H}) \leq \beta \mu_{\rm H}\}, \\ \mathcal{N}_2(\beta) & := & \{(x_{\rm H}, y_{\rm H}) \in \mathcal{K}_{\rm H} \times \mathcal{K}_{\rm H} \mid d_2(x_{\rm H}, y_{\rm H}) \leq \beta \mu_{\rm H}\}, \\ \mathcal{N}_{-\infty}(\beta) & := & \{(x_{\rm H}, y_{\rm H}) \in \mathcal{K}_{\rm H} \times \mathcal{K}_{\rm H} \mid d_{-\infty}(x_{\rm H}, y_{\rm H}) \leq \beta \mu_{\rm H}\} \end{array} \tag{27}$$

where $\beta \in (0,1)$, $w_{\rm H} = Q_{x_{\rm H}^{1/2}} y_{\rm H}$ and

$$\begin{array}{rcl} \mu_{\rm H} & := & \langle x_{\rm H}, y_{\rm H} \rangle / (r+1) \\ d_{\rm F}(x_{\rm H}, y_{\rm H}) & := & \| Q_{x_{\rm H}^{1/2}} y_{\rm H} - \mu_{\rm H} e_{\rm H} \|_{\rm F} = \sqrt{\displaystyle\sum_{i=1}^{r+1} (\lambda_i(w_{\rm H}) - \mu_{\rm H})^2} \\ d_2(x_{\rm H}, y_{\rm H}) & := & \| Q_{x_{\rm H}^{1/2}} y_{\rm H} - \mu_{\rm H} e_{\rm H} \|_2 \\ & = & \max \{ |\lambda_i(w_{\rm H}) - \mu_{\rm H}| \;\; (i = 1, \ldots, r+1) \} \\ & = & \max \{ \lambda_{\max}(w_{\rm H}) - \mu_{\rm H}, \; \mu_{\rm H} - \lambda_{\min}(w_{\rm H}) \}, \\ d_{-\infty}(x_{\rm H}, y_{\rm H}) & := & \mu_{\rm H} - \lambda_{\min}(w_{\rm H}). \end{array}$$

Since the inclusive relation $\mathcal{N}_{\rm F}(\beta) \subseteq \mathcal{N}_2(\beta) \subseteq \mathcal{N}_{-\infty}(\beta)$ holds for any $\beta \in (0,1)$ (see Proposition 29 of [107]), we call the algorithms using $\mathcal{N}_{\rm F}(\beta)$, $\mathcal{N}_2(\beta)$ and $\mathcal{N}_{-\infty}(\beta)$ the *short-step algorithm*, the *semi-long-step algorithm* and the *long-step algorithm*, respectively.

Under Assumption 3.5, we obtain the following results by analogous discussions as in [107] and in [104] (see Corollary 7.3 of [127]).

Theorem 3.6 (Iteration bounds of homogeneous algorithms)**.** *Suppose that $\psi : \mathcal{K} \to \mathcal{V}$ satisfies Assumption 3.5, and we use the NT, xy or yx method for determining the search direction. Then the number of iterations of each homogeneous algorithm is bounded as follows:*

	NT method	*xy or yx method*
Short-step using $\mathcal{N}_{\mathrm{F}}(\beta)$	$\mathcal{O}\left(\sqrt{r}(1+\sqrt{r}\,\theta)\log\epsilon^{-1}\right)$	$\mathcal{O}\left(\sqrt{r}(1+\sqrt{r}\,\theta)\log\epsilon^{-1}\right)$
Semi-long-step using $\mathcal{N}_2(\beta)$	$\mathcal{O}\left(r(1+\sqrt{r}\,\theta)\log\epsilon^{-1}\right)$	$\mathcal{O}\left(r(1+\sqrt{r}\,\theta)\log\epsilon^{-1}\right)$
Long-step using $\mathcal{N}_{-\infty}(\beta)$	$\mathcal{O}\left(r(1+\sqrt{r}\,\theta)\log\epsilon^{-1}\right)$	$\mathcal{O}\left(r^{1.5}(1+\sqrt{r}\,\theta)\log\epsilon^{-1}\right)$

Here $\epsilon > 0$ is the accuracy parameter and $\theta \geq 0$ is the scaled Lipschitz parameter appearing in Assumption 3.5, and $\theta = 0$ holds if ψ is affine.

3.2.3 Other infeasible interior point algorithms

In [75], another infeasible interior point algorithm has been provided for solving the standard linear SCCP. The algorithm is essentially a combination of the standard interior-point methods and the smoothing Newton methods described in the next section, and can be regarded as an extension of the algorithm proposed by Xu and Burke [124]. The algorithm is based on the Chen-Harker-Kanzow-Smale (CHKS) smoothing function $\Psi_\mu : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ (see Section 4.1) and the following target system

$$\begin{pmatrix} y - Px - a \\ \Psi_\mu(x, y) \end{pmatrix} = 0 \tag{28}$$

is considered with some parameter $\mu > 0$. Each iteration employs an approximate Newton method with the Nesterov-Todd direction and the semi-long step neighborhood $\mathcal{N}_2(\beta)$ in (27) . The algorithm enjoys the following theorem (see Theorem 4.7 of [75]).

Theorem 3.7 (Iteration bounds of combined algorithms)**.** *Let $\epsilon > 0$ be a given accuracy parameter. If the standard linear SCCP has a solution then the algorithm is polynomial-time convergent and the iteration complexity is $\mathcal{O}(\sqrt{r}\log\epsilon^{-1})$ for the feasible version of the algorithm and $\mathcal{O}(r\log\epsilon^{-1})$ for the infeasible version of the algorithm.*

The subsequent iterated point by the combined algorithm automatically remains in the neighborhood of central path for the full Newton step, which is a merit of the algorithm.

3.2.4 Local convergence properties

Very few studies on local convergence properties of interior point algorithms for the SCCP are yet available while a description on this issue can be found in [101]. On the other hand, many studies have been done on the monotone implicit linear SDCP (4) (called as the monotone implicit SDLCP).

There are two directions in the study of local convergence of interior point algorithms for the monotone implicit SDLCP (4).

The first approach is to analyze the generated sequence by the infeasible interior point algorithms employing several types of search directions and neighborhoods (see, e.g., [98, 55, 56, 74]).

The second approach is to analyze the infeasible interior point trajectory (central path) [102] or a neighboring *off-central path* of the monotone implicit SDLCP (4) with $m = 0$ (i.e., the variable z has vanished) [109, 110, 111] which is defined by the following ordinary differential equations (ODEs)

$$\begin{array}{l} H_P(XV + UY) = \frac{1}{\mu} H_P(XY), \\ P(U) + Q(V) = O, \\ (X(1), Y(1)) = (X_0, Y_0) \end{array} \tag{29}$$

where $\mu > 0$ is a parameter, X and Y are functions of μ, U and V are derivatives of X and Y, respectively, $X_0 \in \operatorname{int} \mathcal{K}$, $Y_0 \in \operatorname{int} \mathcal{K}$, $P(X_0) + Q(Y_0) - a = 0$ and

$$H_P(U) := \frac{1}{2} \left(PUP^{-1} + (PUP^{-1})^T \right).$$

While the above ODEs define a feasible off-central path, infeasible off central paths have been also analyzed in [112].

At present, the following condition is indispensable to discuss superlinear or quadratic convergence of the infeasible interior point algorithms or analyticity of off-central paths.

Condition 3.8 (Strictly complementarity solution)**.** *There exists a strictly complementary solution* (X_*, Y_*) *which is a solution of the SDCP satisfying*

$$X_* + Y_* \in \operatorname{int} \mathcal{K} = \mathcal{S}^n_{++} := \{X \in \mathcal{S}^n \mid X \succ O\}$$

where $X \succ O$ *denotes that* X *is a positive definite matrix.*

4 Merit or smoothing function methods for the SCCP

Another general class of algorithms is so called merit or smoothing function methods which have been proposed especially for solving nonlinear complementarity problems. Let us consider the monotone vertical SCCP (5) satisfying (7). Here we introduce some important notions in the study of the merit function methods for the vertical SCCP.

Definition 4.1 (Merit function for the SCCP)**.** *A function* $f : \mathcal{V} \to [0, \infty)$ *is said to be a merit function on* $\mathcal{U}$ *for the SCCP if it satisfies*

$$x \text{ is a solution of the SCCP} \iff f(x) = 0$$

on a set $\mathcal{U}$ *(typically* $\mathcal{U} = \mathcal{V}$ *or* $\mathcal{U} = G^{-1}(\mathcal{K})$*) .*

By using the merit function, we can reformulate the SCCP as the following minimization problem

$$\inf\{f(x) \mid x \in \mathcal{U}\}.$$

The merit function methods are to apply a feasible descent method to solve the above minimization problem.

A desirable property for a merit function is that the level sets are bounded. Another desirable property is that it gives an error bound defined as follows.

Definition 4.2 (Error bound for the SCCP)**.** *Let S be the solution set of the SCCP and dist(x,S) denote the distance between $x \in \mathcal{V}$ and the set S. A function $f : \mathcal{V} \to \Re$ is said to be a local error bound for the SCCP if there exist three constants $\tau > 0$, $\eta \in (0,1]$ and $\epsilon > 0$ such that*

$$\mathrm{dist}(x,S) \le \tau f(x)^{\eta} \tag{30}$$

holds for any $x \in S$ with $f(x) \le \epsilon$.

The merit function is closely related to the following C-function (see, e.g., [25] in the case of the NCP).

Definition 4.3 (C-function for the SCCP)**.** *The function $\Phi : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ satisfying*

$$\langle x, y \rangle = 0,\ x \in \mathcal{K},\ y \in \mathcal{K} \iff \Phi(x,y) = 0 \tag{31}$$

is said to be a C-function for the SCCP.

It is clear that if $\Phi : \mathcal{V} \times \mathcal{V} \to [0,\infty)$ is a C-function for the SCCP, then

$$f(x) := \|\Phi(F(x), G(x))\|^2$$

is a merit function for the vertical SCCP (5). For a C-function Φ, the equation $\Phi(x,y) = 0$ is called the *reformation equation* of the complementarity condition of the SCCP.

The C-function Φ is typically nonsmooth but semismooth. In what follows, we introduce the notion of semismoothness of a function according to a fundamental paper by Sun and Sun [117] where some special functions (Löwner operator described in Section 4.1) over symmetric cones have been studied intensively.

Let $\mathcal{U}$ and $\mathcal{V}$ be two finite dimensional inner product space over the field $\Re$. Let $\mathcal{W} \subseteq \mathcal{U}$ be an open subset of $\mathcal{U}$ and $\Phi : \mathcal{W} \to \mathcal{V}$ be a locally Lipschitz continuous function on the open set $\mathcal{W}$. By Rademachers theorem, Φ is almost everywhere (in the sense of Lebesgue measure) differentiable (in the sense of Fréchet) in $\mathcal{W}$. Let $\mathcal{D}_\Phi$ be the set of points in $\mathcal{W}$ where Φ is differentiable. Let $\Phi'(x)$, which is a linear mapping from $\mathcal{U}$ to $\mathcal{V}$, denote the derivative of Φ at $x \in \mathcal{D}_\Phi$. Then, the *B-subdifferential* of Φ at $x \in \mathcal{W}$, denoted by $\partial_{\mathrm{B}}\Phi(x)$, is the set of $\{\lim_{k\to\infty} \Phi'(x_k)\}$, where $\{x_k\} \in \mathcal{D}_\Phi$ is a sequence converging to x. *Clarke's generalized Jacobian* of Φ at x is the convex hull of $\partial_{\mathrm{B}}\Phi(x)$ (see [23]), i.e., $\partial\Phi(x) := \mathrm{conv}\{\partial_{\mathrm{B}}\Phi(x)\}$. If S is any set of Lebesgue measure zero in $\mathcal{U}$, then

$$\partial\Phi(x) = \mathrm{conv}\left\{ \lim_{k\to\infty} \Phi'(x_k) \mid x_k \to x,\ x_k \in \mathcal{D}_\Phi,\ x_k \notin S \right\} \tag{32}$$

(see Theorem 2.5.1 in [23]). Semismoothness was originally introduced by Mifflin [81] for functionals, and was used to analyze the convergence of bundle type methods for nondifferentiable optimization problems [82].

Definition 4.4 (Semismoothness)**.** *Let $\Phi : \mathcal{W} \subseteq \mathcal{U} \to \mathcal{V}$ be a locally Lipschitz continuous function on the open set $\mathcal{W}$. Let us consider the following conditions for a fixed $x \in \mathcal{W}$.*

(i) Φ *is directionally differentiable at x.*

(ii) *For any $y \to x$ and $V_y \in \partial\Phi(y)$ which depends on y,*

$$\Phi(y) - \Phi(x) - V_y(y - x) = o(\|y - x\|).$$

(iii) *There exists $\rho > 0$ such that for any $y \to x$ and $V_y \in \partial\Phi(y)$ which depends on y,*

$$\Phi(y) - \Phi(x) - V_y(y - x) = \mathcal{O}(\|y - x\|^{1+\rho}).$$

If Φ satisfies (i) and (ii) at x then Φ is said to be semismooth *at x.*

If Φ is semismooth and satisfies (iii) at x then Φ is said to be ρ-order semismooth *at x.*

If Φ is 1-order semismooth at x then Φ is said to be strongly semismooth *at x.*

If Φ is semismooth (respectively, ρ-order semismooth) at any point of $\mathcal{Q} \subseteq \mathcal{W}$ then Φ is said to be semismooth (respectively, ρ-order semismooth) on the set $\mathcal{Q}$.

Many approaches have been proposed for approximating the C-function Φ by a so-called *smoothing function* Φ_μ introducing a *smoothing parameter* $\mu > 0$ in order to apply Newton-type methods to the function. By using a C-function or a smoothing function, the vertical SCCP (5) can be reformulated or approximated as the problem to find a solution of the system

$$\Phi(F(x), G(x)) = 0 \quad \text{or} \quad \Phi_\mu(F(x), G(x)) = 0.$$

While the smoothness of the function suggests the Newton method to solve the system, the semismoothness of the function allows us to adopt a Newton-type method, so-called *semismooth Newton method*, which is a significant property of the semismooth functions (see Chapter 7 of [25] for the case $\mathcal{K} = \Re^n_+$). We call such methods *smoothing function methods*. The smoothing parameter $\mu > 0$ is often dealt as a variable in smoothing function methods.

The merit or smoothing function methods for the SCCP were first proposed for the SDCP as a special case, followed by the SOCCP and the SCCP. We will give a brief survey on the studies chronologically according to research progress. Since the SCCP includes the SDCP and the SOCCP, the results obtained for the SCCP hold for the SDCP and the SOCCP as well.

4.1 Merit or smoothing function methods for the SDCP

A pioneer work on the study of merit function method for solving the SDCP has been done by Tseng [122]. For a while, we consider that $\mathcal{S}^n$ is the set of all $n \times n$ symmetric matrices and $\mathcal{K}$ is the positive semidefinite cone. The inner product and the norm are given by

$$\langle X, Y \rangle := \mathrm{Tr}\,(X^T Y), \quad \|X\| := \sqrt{\langle X, X \rangle}.$$

It has been shown in [122] that each of the following functions is a merit function for the vertical SDCP (5).

Natural residual function (Proposition 2.1 of [122]) Let the function $[\cdot]_+$ denote the orthogonal projection onto $\mathcal{S}^n_+$, i.e.,

$$[X]_+ = \arg\min_{W \in \mathcal{S}^n_+} \|X - W\| \tag{33}$$

and define the natural residual function on $\mathcal{S}^n$ as

$$\Phi^{\mathrm{NR}}(A, B) := B - [B - A]_+ \tag{34}$$

Then the function

$$f(X) := \|\Phi^{\mathrm{NR}}(F(X), G(X))\|^2 \tag{35}$$

is a merit function on $\mathcal{S}^n$ for the vertical SDCP (5). The natural residual function has been studied in [24, 77, 73, 97] for the NCP.

Fischer-Burmeister function (Proposition 6.1 of [122]) Let $\phi^{\mathrm{FB}} : \mathcal{S}^n \times \mathcal{S}^n \to \mathcal{S}^n$ be the Fischer-Burmeister function on $\mathcal{S}^n$ given by

$$\Phi^{\mathrm{FB}}(A, B) := (A + B) - (A^2 + B^2)^{1/2}. \tag{36}$$

Then the function

$$f(X) := \frac{1}{2}\|\Phi^{\mathrm{FB}}(F(X), G(X))\|^2 \tag{37}$$

is a merit function on $\mathcal{S}^n$ for the vertical SDCP (5). The Fischer-Burmeister function has been proposed by Fischer [30, 31, 32] for the NCP.

The fact that the functions (35) and (37) are merit functions of the vertical SDCP (5) implies that $\Phi^{\mathrm{NR}}(x, y)$ and $\Phi^{\mathrm{FB}}(x, y)$ are C-function for the vertical SDCP (5).

For each of the above functions f, Tseng [122] derived conditions on F and G for f to be convex and/or differentiable, and for the stationary point of f to be a solution of the SDCP.

On the other hand, Chen and Tseng [17] introduced a wide class of functions so called Chen and Mangasarian smoothing functions [9, 10, 123].

Chen and Mangasarian smoothing functions Let $g : \Re \to \Re$ be a univariate function satisfying

$$\lim_{\tau \to -\infty} g(\tau) = 0, \ \lim_{\tau \to \infty} g(\tau) - \tau = 0, \quad 0 < g'(\tau) < 1. \tag{38}$$

For each symmetric matrix X having the spectral decomposition

$$X = \sum_{i=1}^{r} \lambda_i(X) u_i u_i^T \tag{39}$$

where $\lambda_i(X)$ and u_i are the eigenvalues and the corresponding eigenvectors of X, define the function $g_{\mathcal{S}^n} : \mathcal{S}^n \to \mathcal{S}^n$ as

$$g_{\mathcal{S}^n}(X) := \sum_{i=1}^{r} g(\lambda_i(X)) u_i u_i^T. \tag{40}$$

Then the function

$$\Phi_\mu(X, Y) := X - \mu g_{\mathcal{S}^n}((X - Y)/\mu) \tag{41}$$

with a positive parameter $\mu > 0$ is a function in the class of Chen and Mangasarian smoothing functions.

The function Φ_μ can be regarded as a smoothed approximation to the natural residual function Φ^{NR} by a smoothing parameter $\mu > 0$. The function Φ_μ is continuously differentiable having the property that

$$\Phi_\mu(A, B) \to 0 \text{ and } (A, B, \mu) \to (X, Y, 0) \Longrightarrow X \in \mathcal{S}^n_+, \ Y \in \mathcal{S}^n_+, \langle X, Y \rangle = 0.$$

If we choose

$$g(\tau) := ((\tau^2 + 4)^{1/2} + \tau)/2 \tag{42}$$

then the function Φ_μ is an extension of Chen and Harker [7, 8], Kanzow [50, 51] and Smale [113] (CHKS) smoothing function for the NCP and if we choose

$$g(\tau) := \ln(e^\tau + 1)$$

then the function Φ_μ is an extension of Chen and Mangasarian [9, 10] smoothing function for the NCP. In [17], the authors provided a smoothing continuation method using the the function Φ_μ and studied various issues on existence of Newton directions, boundedness of iterates, global convergence, and local superlinear convergence. They also reported a preliminary numerical experience on semidefinite linear programs.

Note that the function $g_{\mathcal{S}^n} : \mathcal{S}^n \to \mathcal{S}^n$ in (40) is constructed from a univariate function $g : \Re \to \Re$. Such a function has been introduced by Löwner [71] in 1934, and we call the function $g_{\mathcal{S}^n}$ the Löwner operator. See also [117, 58]. It is known that the orthogonal projection (33) onto $\mathcal{S}^n_+$ is given by

$$[X]_+ = \sum_{i=1}^{n} [\lambda_i(X)]_+ u_i u_i^T \tag{43}$$

where X has the spectral decomposition (39) and $[\cdot]_+ : \Re \to \Re$ is the plus function $[t]_+ := \max\{0, t\}$ [122]. This fact implies that $[\cdot]_+$ is also a Löwner operator.

Several papers have been focused on the properties of Löwner operators. Sun and Sun [114] showed that the projection function $[\cdot]_+ : \mathcal{S}^n \to \mathcal{S}^n$ and the function Φ^{NR} in (34) are strongly semismooth on $\mathcal{S}^n$(see Definition 4.4). Chen and Tseng [18] extended the results and investigated that the Löwner operator $g_{\mathcal{S}^n}$ inherits from g the properties of continuity, (local) Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, as well as ρ-order semismoothness in the case of the SDCP. These results have been extended to the function $g_{\mathcal{V}}$ over the Euclidean Jordan algebra by Sun and Sun [117], which we will describe in a little more detail in Section 4.3.

For the Fischer-Burmeister function $\Phi^{\mathrm{FB}} : \mathcal{S}^n \times \mathcal{S}^n \to \mathcal{S}^n$ in (36), Sun and Sun [116] showed that Φ^{FB} is globally Lipschitz continuous, directionally differentiable, and strongly semismooth.

See [122] and [125] for other merit functions proposed for the SDCP. Further developments also can be seen in [6, 47, 106, 130], etc. Meanwhile, besides the paper [17], there are many papers on smoothing continuation methods for the SDCP, e.g., see [18, 46, 52, 115, 116].

4.2 Merit or smoothing function methods for the SOCCP

A first study on smoothing function methods for solving the SOCCP has been provided by Fukushima, Luo and Tseng [33]. For the implicit SOCCP (3), the authors introduced the class of Chen and Mangasarian smoothing functions, studied the Lipschitzian and differential properties of the functions, and derived computable formulas for the functions and their Jacobians. They also showed the existence and uniqueness of the Newton direction when the mapping F is monotone.

In [19], the authors introduced a C-function for the standard SOCCP (2) and showed that the squared C-function is strongly semismooth. They also reported numerical results of the squared smoothing Newton algorithms.

Chen, Sun and Sun [11] showed that the Löwner operator $g_{\mathcal{V}}$ for the standard SCCP (2) inherits from g the properties of continuity, (local) Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, as well as ρ-order semismoothness.

The SOCCP has a simple structure, i.e., the rank r of the second order cone is always $r = 2$. This may be a reason why the SOCCP has many applications, e.g, the *three dimensional quasi-static frictional contact* [49] and the *robust Nash equilibria* [44, 90].

See [12, 13, 14, 15, 16, 5, 26, 40, 91, 96, 94, 99, 118, 129] for many other merit function methods for the SOCCP, [45, 116] for smoothing continuation methods for the SOCCP and [60] for the solution set structure of the monotone SOCCP.

4.3 Merit or smoothing function methods for the SCCP

The merit or smoothing function methods for the SCCP have become a very active research area in recent few years. Most of the results for the SDCP and the SOCCP have been extended to the SCCP.

A first C-function for the SCCP has been proposed by Gowda [36] where some **P**-properties for the SCCP have been discussed (see Section 5). The following proposition ensures that the natural residual function $\Phi^{\mathrm{NR}}(x, y) := x - [x - y]_+$ and the Fischer-Burmeister function $\Phi^{\mathrm{FB}}(x, y) := x + y - \sqrt{x^2 + y^2}$ are C-functions for the SCCP (see Proposition 6 of [36]).

Proposition 4.5 (C-functions for the SCCP)**.** *For $(x, y) \in \mathcal{V}$, the following conditions are equivalent.*

(a) $x - [x - y]_+ = 0$

(b) $(x, y) \in \mathcal{K} \times \mathcal{K}$ *and* $\langle x, y\rangle = 0$.

(c) $(x, y) \in \mathcal{K} \times \mathcal{K}$ *and* $x \circ y = 0$.

(d) $x + y - \sqrt{x^2 + y^2} = 0$

(e) $x + y \in \mathcal{K}$ *and* $x \circ y = 0$.

Here $[x]_+$ denotes the orthogonal projection onto $\mathcal{K}$, i.e., $[x]_+ = \arg\min_{w\in\mathcal{K}} \|x - w\|$.

Suppose that $x \in \mathcal{V}$ has the spectral decomposition $x = \sum_{i=1}^{r} \lambda_i(x)c_i$ (see Theorem 2.1). For a given univariate function $\phi : \Re \to \Re$, we define the vector-valued function $\phi_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ associated with the Euclidean Jordan algebra $(\mathcal{V}, \circ, \langle\cdot, \cdot\rangle)$ by

$$\phi_{\mathcal{V}}(x) := \sum_{i=1}^{r} \phi(\lambda_i(x))c_i. \tag{44}$$

The function $\phi_{\mathcal{V}}$ is the (extended) Löwner operator for which Korányi [65] extended Löwner's results on the monotonicity of $\phi_{\mathcal{S}^n}$ to $\phi_{\mathcal{V}}$. For an example, the projection function $[\cdot]_+ : \mathcal{V} \to \mathcal{V}$ can be represented by

$$[x]_+ = \sum_{i=1}^{r} [\lambda_i(x)]_+ c_i$$

where $[\cdot]_+ : \Re \to \Re$ is the plus function $[t]_+ := \max\{0, t\}$ (see [36]).

The fundamental paper of Sun and Sun [117] gives basic properties of Löwner operators over $(\mathcal{V}, \circ, \langle\cdot, \cdot\rangle)$ on differentiability and semismoothness. Here we refer the following important results (see Theorem 3.2, Propositions 3.3, 3,4 and Theorem 3.3 of [117]).

Theorem 4.6 (Differentiability of $\phi_{\mathcal{V}}$)**.** *Let $x = \sum_{i=1}^{r} \lambda_i(x)c_i$. The function $\phi_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is differentiable (respectively, continuously differentiable) at x if and only if for each $i \in \{1, 2, \ldots, r\}$, $\phi : \Re \to \Re$ is differentiable (continuously differentiable) at $\lambda_i(x)$.*

Theorem 4.7 (Semismoothness of $\phi_{\mathcal{V}}$)**. (i)** *The projection $[\cdot]_+ : \mathcal{V} \to \mathcal{V}$ is strongly semismooth on $\mathcal{V}$.*

(ii) *For $\epsilon \in \Re$, define $\phi^{\mu} : \Re \to \Re$ by*

$$\phi^{\mu}(t) := \sqrt{t^2 + \mu^2}.$$

The corresponding Löwner operator is given by

$$\phi^{\mu}_{\mathcal{V}}(x) := \sum_{i=1}^{r} \sqrt{\lambda_i(x)^2 + \mu^2}c_i = \sqrt{x^2 + \mu^2 e}$$

which is a smoothed approximation to the absolute value function $|x| := \sqrt{x^2}$ (see also (41) and (42)). Let $\psi(\mu, x) := \phi^{\mu}_{\mathcal{V}}(x)$. Then $\psi : \Re \times \mathcal{V} \to \mathcal{V}$ is continuously differentiable at (μ, x) if $\mu \neq 0$ and is strongly semismooth at $(0, x)$, $x \in \mathcal{V}$.

(iii) *Let $\rho \in (0, 1]$ be a constant and $x = \sum_{i=1}^{r} \lambda_i(x)c_i$. The function $\phi_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is semismooth (respectively, ρ-order semismooth) if and only if for each $i \in \{1, 2, \ldots, r\}$, $\phi : \Re \to \Re$ is semismooth (ρ-order semismooth) at $\lambda_i(x)$.*

Kong, Tunçel and Xiu [58] investigated the connection between the monotonicity of the function ϕ and its corresponding Löwner operator. For a given real interval (a, b) with $a < b$ $(a, b \in \Re \cup \{-\infty\} \cup \{\infty\})$, denote by $\mathcal{V}(a, b)$ the set of all $x \in \mathcal{V}$ such that $x - ae, be - x \in \operatorname{int} K$. They showed the following properties (see Theorem 3 of [58]).

Theorem 4.8 (Convexity of $\phi_{\mathcal{V}}$)**.** *Let g be a locally Lipschitz function from (a, b) into $\Re$. $\phi_{\mathcal{V}}$ is monotone (respectively, strictly monotone, strongly monotone) on $\mathcal{V}(a, b)$ if and only if ϕ is monotone (respectively, strictly monotone, strongly monotone) on (a, b).*

Many studies have been conducted on merit or smoothing function methods for the SCCP and some of them are still continuing even today. The subjects dealt in these studies are the natural residual function [63], the Fischer-Burmeister (smoothing) function [4, 59, 70], Chen-Mangasarian smoothing functions [22, 61, 48], other merit functions [42, 57, 62, 66, 67, 68, 69, 92, 95], and smoothing continuation methods [48, 61, 89, 21, 22, 126], etc.

5 Properties of the SCCP

In the discussions so far, we assume that the SCCP is monotone. In this chapter, we observe other various properties of the SCCP. We address the following two SCCPs, LCP(L, q) and NLCP(ψ, q), which are alternate representations of the standard linear or nonlinear SCCP (2).

$$\left|\begin{array}{lll} \text{LCP}(L, q) & \text{Find} & x \in \mathcal{K} \\ & \text{s.t.} & x \in \mathcal{K},\ Lx + q \in \mathcal{K},\ \langle x, Lx + q\rangle = 0 \end{array}\right. \tag{45}$$

where $q \in \mathcal{V}$ and $L : \mathcal{V} \to \mathcal{V}$ is a linear operator.

$$\left|\begin{array}{lll} \text{NLCP}(\psi, q) & \text{Find} & x \in \mathcal{K} \\ & \text{s.t.} & x \in \mathcal{K},\ \psi(x) + q \in \mathcal{K},\ \langle x, \psi(x) + q\rangle = 0 \end{array}\right. \tag{46}$$

where $q \in \mathcal{V}$ and $\psi : \mathcal{V} \to \mathcal{V}$ is a (nonlinear) continuous function.

The linear operator L and $q \in V$ determine the property of LCP(L, q). Gowda, Sznajder and Tao [36] focused their attention on the **P**-property of the linear operator L. If we restrict $\mathcal{V} = \Re^n$ and $\mathcal{K} = \Re^n_+$ then the **P**-property of the linear operator L can be characterized as follows (see [3, 24, 25], etc.).

Proposition 5.1 (**P**-property of the LCP over $\Re^n_+$)**.** *Let $\mathcal{V} = \Re^n$ and $\mathcal{K} = \Re^n_+$. Then the following properties of the linear operator $L : \Re^n \to \Re^n$ are equivalent.*

(i) *All principal minors of L are positive.*

(ii)

$$x \in \Re^n, \ x \circ Lx \le 0 \implies x = 0$$

where $a \circ b$ denotes the componentwise product for $a, b \in \Re^n$.

(iii)

$$x \wedge Lx \le 0 \le x \vee Lx \implies x = 0$$

where $a \wedge b$ and $a \vee b$ denote the componentwise minimum and maximum of $a, b \in \Re^n$, respectively.

(iv) *For all $q \in \Re^n$, there exists a unique $x \in \Re^n$ such that*

$$x \ge 0, \ Lx + q \ge 0, \ \langle x, Lx + q\rangle = 0.$$

(v) *The function $F(x) := L[x]_+ + x - [x]_+$ is invertible in a neighborhood of zero.*

(vi) *The function $F(x) := L[x]_+ + x - [x]_+$ is invertible in a neighborhood of zero with Lipschitzian inverse.*

Note that the function $F(x) := L[x]_+ + x - [x]_+$ is called the *normal map* of the SCCP and known as a C-function for the SCCP (see [21] and also see Section 1.5 of [25] in the case of the NCP). In contrast, the above properties are different for the SCCP in general. How should we define the **P**-property for the SCCP? Gowda, Sznajder and Tao [36] introduced the following properties (i)-(viii) of the linear operator $L : \mathcal{V} \to \mathcal{V}$ in the setting of a Euclidean Jordan algebra.

Definition 5.2 (**P**-properties of $L : V \to V$)**.** *The linear operator $L : \mathcal{V} \to \mathcal{V}$ is said to be (to have)*

(i) **monotone** *(respectively,* **strictly monotone***) if $\langle Lx, x\rangle \ge 0$ (respectively, $\langle Lx, x\rangle > 0$) for all $0 \ne x \in V$,*

(ii) *the* **order P***-property if*

$$x \sqcap Lx \in -K, \ x \sqcup Lx \in K \implies x = 0,$$

where $x \sqcap y := x - [x - y]_+$ and $x \sqcup y := y + [x - y]_+$,

(iii) *the* **Jordan P***-property if*

$$x \circ Lx \in -K \implies x = 0,$$

(iv) *the* **P***-property if*

$$\left.\begin{array}{l} x \text{ and } Lx \text{ operator commute} \\ x \circ Lx \in -K \end{array}\right\} \implies x = 0,$$

(v) *the* **Globally Uniquely Solvable (GUS)** *if any $q \in \mathcal{V}$, LCP(L, q) has a unique solution.*

(vi) *the* **Cross Commutative property (Cross Commutative)** *if for any $q \in \mathcal{V}$ and for any two solution x_1 and x_2 of LCP(L, q), x_1 operator commutes with $y_2 = Lx_2 + q$ and x_2 operator commutes with $y_1 = Lx_1 + q$.*

(vii) *the* **Lipschitzian GUS**-*property if the normal map $F(x) := L[x]_+ + x - [x]_+$ is a homeomorphism of $\mathcal{V}$ and the inverse of F is Lipschitzian.*

(viii) *the* **positive principal minor (positive PM)** *if all principal minors of L are positive.*

Gowda, Sznajder and Tao [36] showed the following implications (see Theorems 11, 12, 14, 17 and 23 and Examples 1.3 and 2.3 of [36]).

Proposition 5.3 (Implications among the properties on L)**.**

(i) Strictly monotonicity $\Longrightarrow$ Order P $\Longrightarrow$ Jordan P.

(ii) *If L has* **P**-*property then the every real eigenvalue of L is positive.*

(iii) *If L has* **P**-*property then for any $q \in \mathcal{V}$, LCP(L, q) has a nonempty compact solution set.*

(iv) GUS = P + Cross Commutative.

(v) Strictly monotone $\Longrightarrow$ Lipschitzian GUS $\Longrightarrow$ positive PM.

(vi) GUS $\not\Longleftarrow$ Lipschitzian GUS.

(vii) Order P $\not\Longleftarrow$ strong monotonicity.

(viii) *If $\mathcal{K}$ is polyhedral then,* **order P = Jordan P = P = GUS = positive PM**.

Moreover, Tao and Gowda [119] extended the above results to NLCP(ψ, q) introducing the following properties.

Definition 5.4 (**P**-properties of $\psi : V \to V$)**.** *The continuous function $\psi : \mathcal{V} \to \mathcal{V}$ is said to be (to have)*

(i) monotone *if for all $x, y \in \mathcal{V}$, $\langle x - y, \psi(x) - \psi(y)\rangle \geq 0$.*

(ii) strictly monotone *if for all $x \neq y \in \mathcal{V}$, $\langle x - y, \psi(x) - \psi(y)\rangle > 0$.*

(iii) strongly monotone *if there is an $\alpha > 0$ such that for all $x, y \in \mathcal{V}$,*

$$\langle x - y, \psi(x) - \psi(y)\rangle \geq \alpha \|x - y\|^2,$$

(iv) *the* **order P**-*property if*

$$(x - y) \sqcap (\psi(x) - \psi(y)) \in -K, (x - y) \sqcup (\psi(x) - \psi(y)) \in K \implies x = y,$$

(v) *the* **Jordan P**-*property if*

$$(x - y) \circ (\psi(x) - \psi(y)) \in -K \implies x = y,$$

(vi) *the* **P**-*property if*

$$\left.\begin{array}{l} x - y \text{ and } \psi(x) - \psi(y) \text{ operator commute} \\ (x - y) \circ (\psi(x) - \psi(y)) \in -K \end{array}\right\} \implies x = y,$$

(vii) *the* **uniform Jordan P**-*property if there is an* $\alpha > 0$ *such that for all* $x, y \in \mathcal{V}$,

$$\lambda_{\max}\left((x-y) \circ (\psi(x) - \psi(y))\right) \geq \alpha \|x-y\|^2,$$

(viii) *the* **uniform P**-*property if there is an* $\alpha > 0$ *such that for all* $x, y \in \mathcal{V}$ *with* $x - y$ *operator commuting with* $\psi(x) - \psi(y)$,

$$\lambda_{\max}\left((x-y) \circ (\psi(x) - \psi(y))\right) \geq \alpha \|x-y\|^2,$$

(ix) *the* $\mathbf{P}_0$-*property if* $\psi(x) + \epsilon x$ *has the* **P**-*property for all* $\epsilon > 0$,

(x) *the* $\mathbf{R}_0$-*property if for any sequence* $\{x_k\}$ *in* $\mathcal{V}$ *with*

$$\|x_k\| \to \infty, \ \liminf_{k\to\infty} \frac{\lambda_{\min}(x_k)}{\|x_k\|} \geq 0, \ \liminf_{k\to\infty} \frac{\lambda_{\min}(\psi(x_k))}{\|x_k\|} \geq 0,$$

we have

$$\liminf_{k\to\infty} \frac{\langle x_k, \psi(x_k)\rangle}{\|x_k\|^2} \geq 0.$$

Here $\lambda_{\max}(x)$ *and* $\lambda_{\min}(x)$ *denote the maximum and the minimum eigenvalues of* x*, respectively.*

Tao and Gowda [119] showed the following implications (see Proposition 3.1, Theorem 3.1, Propositions 3.2 and 3.3, Corollary 3.1 and Theorem 4.1 of [119]).

Proposition 5.5 (Implications among the properties on ψ)**.**

(i) Strong monotonicity $\Longrightarrow$ strict monotonicity $\Longrightarrow$ order P $\Longrightarrow$ Jordan P $\Longrightarrow$ P $\Longrightarrow$ $\mathbf{P}_0$.

(ii) Strong monotonicity $\Longrightarrow$ uniform Jordan P $\Longrightarrow$ uniform P.

(iii) Monotonicity $\Longrightarrow$ $\mathbf{P}_0$.

(iv) *If* $\psi : \mathcal{V} \to \mathcal{V}$ *has the* $\mathbf{P}_0$-*property and for any* $\delta > 0$ *in* $\Re$ *and the set*

$$\{x \in \mathcal{K} \mid x \text{ solves } NLCP(\psi, q), \ \|q\| \leq \delta\} \tag{47}$$

is bounded, then for any $q \in \mathcal{V}$, *NLCP*(ψ, q) *has a nonempty bounded solution set.*

(v) *If* $\psi : \mathcal{V} \to \mathcal{V}$ *has the* $\mathbf{R}_0$-*property, then for any* $\delta > 0$ *in* $\Re$, *the set (47) is bounded,*

(vi) *If* $\psi : \mathcal{V} \to \mathcal{V}$ *has the* $\mathbf{P}_0$ *and* $\mathbf{R}_0$ *property, then for all* $q \in \mathcal{V}$, *the solution set of NLCP*(ψ, q) *is nonempty and bounded. Moreover, there exists an* $x \in \operatorname{int} \mathcal{K}$ *such that* $\psi(x) + q \in \operatorname{int} \mathcal{K}$.

(vii) *If* $\psi : \mathcal{V} \to \mathcal{V}$ *has the* **GUS**-*property, then for any primitive idempotent* $c \in \mathcal{V}$, $\langle \psi(c) - \psi(0), c\rangle \geq 0$.

See also [34, 35] for the case of the SDCP.

Kong, Tunçel and Xiu [64] extended the **P**-property to the linear complementarity problem over *homogeneous cones*. Note that homogeneous cones properly include symmetric cones and if a homogeneous cone is self-dual then it is a symmetric cone.

Tao and Gowda [119] also introduced an approach to construct a relaxation transformation based on the Peirce decomposition of a given $x \in \mathcal{V}$ (see Theorem 2.2). Suppose we are given a Jordan frame

$\{c_1, c_2, \ldots, c_r\}$ in $\mathcal{V}$ and a continuous function $\phi : \Re^r \to \Re^r$, $\phi(u) := [\phi_1(u), \phi_2(u), \ldots, \phi_r(u)]$ for $u \in \Re^r$. We can write any $x \in \mathcal{V}$ as the Peirce decomposition

$$x = \sum_{i=1}^{r} x_i c_i + \sum_{i<j} x_{ij}.$$

where $x_i \in \Re$ and $x_{ij} \in \mathcal{V}_{ij} := \{x \in \mathcal{V} \mid x \circ c_i = \frac{1}{2}x = x \circ c_j\}$. Then the *relaxation transformation* $R_\phi : \mathcal{V} \to \mathcal{V}$ is given by

$$R_\phi(x) := \sum_{i=1}^{r} \phi_i([x_1, x_2, \ldots, x_r])c_i + \sum_{i<j} x_{ij}. \tag{48}$$

The following **P**-properties of the relaxation transformation R_ϕ have been shown (see Propositions 5.1 and 5.2 of [119]).

Proposition 5.6 (**P**-properties of the relaxation transformation R_ϕ)**.** *The following statements are equivalent.*

(a) *ϕ is a* **P***-function.*

(b) *R_ϕ has the* **order P***-property.*

(c) *R_ϕ has the* **Jordan P***-property.*

(d) *R_ϕ has the* **P***-property.*

Proposition 5.7 (NLCP(R_ϕ, q) having a nonempty and bounded solution set)**.** *If $\phi : \Re^r \to \Re^r$ has the* $\mathbf{P}_0$*- and* $\mathbf{R}_0$*-property then for the relaxation transformation R_ϕ, the set (47) is bounded for any $\delta > 0$. Hence, by (iii) of Proposition 5.5, NLCP(R_ϕ, q) has a nonempty and bounded solution set.*

Proposition 5.7 implies a significance of the relaxation transformation R_ϕ: Only by investigating the vector valued function ϕ, we can find whether the solution set of the associated SCCP is nonempty and bounded.

Lu, Huang and Han [72] stated the differences between the Löwner operator (44) and the relaxation transformation (48) as follows.

> Both the relaxation transformation and the Löwner operator are the generalization of functions defined in the Euclidean vector space. However, they are more difficult to be discussed than those in the Euclidean vector space. The main difficulty from the Löwner operator is that different elements are given by the different Jordan frames; while the main difficulty from the relaxation transformation is that every transformation R_ϕ has an additional item $\sum_{i<j} x_{ij}$ though a common Jordan frame is used. In addition, there are many differences between the relaxation transformation and the Löwner operator. For example, the Löwner operator $\phi_\mathcal{V}(x)$ is defined based on the spectral decomposition of x, and hence, $\phi_\mathcal{V}(x)$ operator commutes with x; while the relaxation transformation R_ϕ is defined based on the Peirce decomposition of x, and it is easy to see that R_ϕ operator does not commute with x because of the existence of an additional item $\sum_{i<j} x_{ij}$.

They investigated some inter connections between ϕ and R_ϕ concerning continuity, differentiability, semismoothness, monotonicity and **P**-properties. They also provided a smoothing method for solving

NLCP(R_ϕ, q) based on the CHKS smoothing function and showed its global convergence. See [37, 38, 39, 120, 121] for more recent results on these functions.

Another property, sufficiency, of the linear operator L in LCP(R_ϕ, q) has been observed in [103].

Definition 5.8 (Sufficient properties of $L : V \to V$). *The linear operator $L : \mathcal{V} \to \mathcal{V}$ is said to have*

(i) *the* **Jordan column-sufficient (Jordan CSU)** *property if*

$$x \circ Lx \in -K \implies x \circ Lx = 0,$$

(ii) *the* **column-sufficient (CSU)** *property if*

$$\left.\begin{array}{l} x \text{ and } Lx \text{ operator commute} \\ x \circ Lx \in -K \end{array}\right\} \implies x \circ Lx = 0,$$

(iii) *the* **Jordan row-sufficient (Jordan RSU)** *property if the adjoint operator L^* of L has the Jordan column-sufficient property,*

(iii) *the* **row-sufficient (RSU)** *property if the adjoint operator L^* of L has the column-sufficient property.*

Then the following implications hold (see Remark (iii), Theorems 3.5, 4.3 of [103]).

Theorem 5.9 (Implications among sufficient properties on L). **(i)** $\mathbf{P} \implies \mathbf{CSU} \implies \mathbf{P}_0$.

(ii) *For a linear operator $L : \mathcal{V} \to \mathcal{V}$, the following statements are equivalent*

- **(a)** *L has the* **CSU** *and* **Cross Commutative** *properties.*
- **(b)** *For any $q \in \mathcal{V}$, the solution set of LCP(L, q) is convex (possibly empty).*

(iii) *For a linear operator $L : \mathcal{V} \to \mathcal{V}$, the following statements are equivalent*

- **(a)** *L has the* **RSU** *property.*
- **(b)** *For any $q \in \mathcal{V}$, if the KKT point (x, u) of the problem*

$$\begin{array}{l} \min \frac{1}{2}\langle x, Lx + L^*x\rangle + \langle q, x\rangle \\ s.t. x \in K,\ Lx + q \in K \end{array}$$

satisfies that $(x - u)$ and $L^*(x - u)$ operator commute, then x solves LCP(L, q).

Suppose that $\mathcal{V}$ is the Cartesian product space $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \mathcal{V}_m$ with its symmetric cone $\mathcal{K} := \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \mathcal{K}_m$. Then the following properties of the linear operator L in LCP(L, q) and the nonlinear function ψ in NLCP(ψ, q) have been introduced in [20, 62, 76, 93], etc.

Definition 5.10 (**Cartesian P** properties of L and ψ). *The linear operator $L : \mathcal{V} \to \mathcal{V}$ is said to have*

(i) *the* **Cartesian P**-*property if for any nonzero $x = (x_1, x_2, \ldots, x_m) \in \mathcal{V}$ with $x_i \in \mathcal{V}_i$, there exists an index $\nu \in \{1, 2, \ldots, m\}$ such that $\langle x_\nu, (Lx)_\nu\rangle > 0$,*

(ii) *the* **Cartesian $\mathbf{P}_0$**-*property if for any nonzero $x = (x_1, x_2, \ldots, x_m) \in \mathcal{V}$ with $x_i \in \mathcal{V}_i$, there exists an index $\nu \in \{1, 2, \ldots, m\}$ such that $x_\nu \neq 0$ and $\langle x_\nu, (Lx)_\nu\rangle \geq 0$,*

(iii) *the* **Cartesian $\mathbf{P}_*(\kappa)$**-*property if for any* $x = (x_1, x_2, \ldots, x_m) \in \mathcal{V}$ *with* $x_i \in \mathcal{V}_i$,

$$(1+4\kappa) \sum_{\nu \in I_+(x)} \langle x_\nu, (Lx)_\nu \rangle + \sum_{\nu \in I_-(x)} \langle x_\nu, (Lx)_\nu \rangle \geq 0$$

where

$$I_+(x) := \{\nu \mid \langle x_\nu, (Lx)_\nu \rangle > 0\}, \quad I_+(x) := \{\nu \mid \langle x_\nu, (Lx)_\nu \rangle < 0\}.$$

The function $\psi : \mathcal{V} \to \mathcal{V}$ *is said to have*

(iv) *the* **uniform Cartesian P**-*property if there exists a constant* α *such that for any* $x, y \in \mathcal{V}$, *there exists an index* $\nu \in \{1, 2, \ldots, m\}$ *such that* $\langle x_\nu - y_\nu, (\psi(x))_\nu - (\psi(y))_\nu \rangle \geq \alpha \|x-y\|^2$,

(iv) *the* **Cartesian P**-*property if for any* $x \neq y \in \mathcal{V}_i$, *there exists an index* $\nu \in \{1, 2, \ldots, m\}$ *such that* $\langle x_\nu - y_\nu, (\psi(x))_\nu - (\psi(y))_\nu \rangle > 0$,

(v) *the* **Cartesian $\mathbf{P}_0$**-*property if for any* $x \neq y \in \mathcal{V}_i$, *there exists an index* $\nu \in \{1, 2, \ldots, m\}$ *such that* $\langle x_\nu - y_\nu, (\psi(x))_\nu - (\psi(y))_\nu \rangle \geq 0$,

(vi) *the* **Cartesian $\mathbf{P}_*(\kappa)$**-*property if for any* $x, y \in \mathcal{V}$,

$$\begin{aligned} (1+4\kappa) & \sum_{\nu \in I_+(x)} \langle x_\nu - y_\nu, (\psi(x))_\nu - (\psi(y))_\nu \rangle \\ & + \sum_{\nu \in I_-(x)} \langle x_\nu - y_\nu, (\psi(x))_\nu - (\psi(y))_\nu \rangle \geq 0 \end{aligned}$$

where

$$\begin{aligned} I_+(x) &:= \{\nu \mid \langle x_\nu - y_\nu, (\psi(x))_\nu - (\psi(y))_\nu \rangle > 0\}, \\ I_+(x) &:= \{\nu \mid \langle x_\nu - y_\nu, (\psi(x))_\nu - (\psi(y))_\nu \rangle < 0\}. \end{aligned}$$

The above properties have been motivated by a complete characterization of Euclidean Jordan algebras (see the chapter by Farid Alizadeh in this handbook or Chapter V of [27]). For LCP(L, q) over the semidefinite cone $\mathcal{K} = \mathcal{S}^n_+$, it has been shown that the following implications

$$\textbf{Strong monotonicity} \implies \textbf{Cartesian P} \implies \begin{cases} \mathbf{P}, \\ \mathbf{GUS}. \end{cases}$$

hold for any $L : \mathcal{V} \to \mathcal{V}$ [20]. In [93], a merit function based on the Fischer-Burmeister function has been proposed for solving NLCP(ψ, q) where $\mathcal{K}$ is the second order cone $\mathcal{L}^n$ and ψ has the **Cartesian $\mathbf{P}_0$**-property. In [76], a theoretical framework of path-following interior point algorithms has been established for solving LCP(L, q) in the setting of a Euclidean Jordan algebra, where L has the **Cartesian $\mathbf{P}_*(\kappa)$**-property. The global convergence and the iteration complexities have been also discussed.

Furthermore, Chua, Lin and Yi [22] (see also [21]) introduced the following property, *uniform nonsingularity*, of the nonlinear operator $\psi : \mathcal{V} \to \mathcal{V}$ in NLCP(ψ, q),

Definition 5.11 (Uniform nonsingularity)**.** *The function* $\psi : \mathcal{V} \to \mathcal{V}$ *is said to be uniformly nonsingular if there exists* $\alpha > 0$ *such that for any* $d_1, d_2, \ldots, d_r \geq 0$ *and* $d_{ij} \geq 0$, *any Jordan frame* $\{c_1, c_2, \ldots, c_r\}$ *and any* $x, y \in \mathcal{V}$,

$$\left\| f(x) - f(y) + \sum_{i=1}^r d_i (x_i - y_i) c_i + \sum_{i<j} d_{ij} (x_{ij} - y_{ij}) \right\| \geq \alpha \|x - y\|.$$

Note that the uniform nonsingularity is an extension of **P**-properties, i.e., if $\mathcal{V} = \Re^n$ the uniform nonsingularity is equivalent to **P**-function property (see Proposition 4.1 of [21]). In [21], a Chen and Mangasarian smoothing method has been proposed for solving NLCP(ψ, q) where ψ is uniformly nonsingular. In [22], the authors showed several implications among the **Cartesian P** properties and uniform nonsingularity and discussed the existence of Newton directions and the boundedness of iterates of some merit function methods. The authors also showed that LCP(L, q) is globally uniquely solvable under the assumption of uniform nonsingularity.

Some geometrical properties on the solution set of the SCCP have been explored in [78, 79]. Let us denote by SOL(L, q) the solution set of LCP(L, q). A nonempty subset $\mathcal{F}$ is said to be a *face* of a closed convex cone, denoted by $\mathcal{K} \trianglelefteq \mathcal{K}$, if $\mathcal{F}$ is a convex cone and

$$x \in \mathcal{K},\ y - x \in \mathcal{K} \text{ and } y \in \mathcal{K} \implies x \in \mathcal{F}.$$

The *complementary face* of $\mathcal{F}$ is defined as

$$\mathcal{F}^{\triangle} := \{y \in \mathcal{K}^* \mid \langle x, y \rangle = 0 \text{ for all } x \in \mathcal{F}.$$

Malik and Mohan [78] introduced the notion of the complementary cone which was originally introduced in [87] for $\mathcal{K} = \Re^n_+$.

Definition 5.12 (Complementary cone of L)**.** *For a given linear operator $L : \mathcal{V} \to \mathcal{V}$, a complementary cone of L corresponding to the face $\mathcal{F}$ of $\mathcal{K}$ is defined as*

$$\mathcal{C}_{\mathcal{F}} := \{y - Lx \mid x \in \mathcal{F},\ y \in \mathcal{F}^{\triangle}\}.$$

The linear complementarity problem LCP(L, q) has a solution if and only if there exists a face $\mathcal{F}$ of $\mathcal{K}$ such that $q \in \mathcal{C}_{\mathcal{F}}$ (see Observation 1 of [78]). Thus the union of all complementary cones is the set of all vectors q for which LCP(L, q) has a solution, Using these observations, the authors characterized the finiteness of the solution set of LCP(L, q).

6 Concluding remarks

In this chapter, we provide a brief survey on the recent developments related to the symmetric cone complementarity problems. By viewing this, we strongly recognize the outstanding contribution of Paul Tseng to the area. We hope that this manuscript will help to leave his achievement for posterity.

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